Section 6.5

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Recall that we defined the definite integral only for functions that are continuous (or, piecewise continuous) functions over closed and bounded intervals $[a,b]$. How do we extend this definition to cover some other cases?

Improper integrals

Type I: Let f be continuous on $[a,\infty)$. We define the improper integral of f over $[a,\infty)$ to be

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$$

Similarly, for a function f which is continuous on $(-\infty, a]$, we define

$$\int_{-\infty}^a f(x)\,dx = \lim_{R\to-\infty} \int_R^a f(x)\,dx$$

If these limits exist, we say that the corresponding improper integral is convergent. Otherwise, ______ divergent.

- $$\int_0^\infty e^{-x}\,dx = \lim_{R\to\infty} \int_0^R e^{-x}\,dx = \lim_{R\to\infty} \left(-e^{-x}\Big|_0^R\right) = \lim_{R\to\infty} \cancel{-e^{-R}}^{\,0} - (-e^0) = 1$$ convergent

- $$\int_{e^e}^\infty \frac{1}{x\ln x \ln\ln x}\,dx = \int_1^\infty \frac{1}{u}\,du = \ln|u|\Big|_1^\infty = \lim_{R\to\infty} \ln|u|\Big|_1^R = \lim_{R\to\infty} \ln|R| - 0$$
  $$= +\infty$$ divergent

$$\boxed{\begin{aligned} u &= \ln\ln x \\ du &= \frac{1}{x\ln x}\,dx \end{aligned}}$$

$$= \lim_{R\to\infty} \int_{e^e}^R \frac{1}{x\ln x \ln x}\,dx = \lim_{R\to\infty} \int_1^{\ln\ln R} \frac{1}{u}\,du = \lim_{R\to\infty} \ln(\ln(\ln R)) - 0$$
$$= +\infty$$

Type II: Let f be continuous on $(a,b]$, which may be possibly unbounded near a. We define the improper integral of f over $(a,b]$ as

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx$$

Similarly, if f is continuous on $[a,b)$ and possibly unbounded near b, we define

$$\int_a^b f(x)\,dx = \lim_{c\to b^-} \int_a^c f(x)\,dx$$

- $$\int_0^{\pi/2} \tan x\,dx = \lim_{c\to \frac{\pi}{2}^-} \int_0^c \tan x\,dx = \lim_{c\to\frac{\pi}{2}^-} -\ln|\cos x|\Big|_0^c = \lim_{c\to\frac{\pi}{2}^-} -\ln(\cos x)\Big|_0^c$$

$$= \lim_{c\to\frac{\pi}{2}^-} \underbrace{-\ln(\underbrace{\cos(c)}_{\to 0})}_{\to +\infty} - (-\ln(1))$$

$$= +\infty \qquad \text{divergent}$$

After "splitting" the interval of integration at a point, if we get these types of improper integrals, then we can define the improper integral over that interval as the sum of improper integrals that are taken on each interval. PROVIDED THAT BOTH OF THESE IMPROPER INTEGRALS CONVERGE!

Example: Let f be continuous $(-\infty,+\infty)$. Then

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{0} f(x)\,dx}_{\text{type I}} + \underbrace{\int_0^{\infty} f(x)\,dx}_{\text{type I}}$$

Example: Find $\displaystyle\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx$

Solution:

$$\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx = \underbrace{\int_0^1 \frac{1}{\sqrt{2x-x^2}}\,dx}_{\text{type II}} + \underbrace{\int_1^2 \frac{1}{\sqrt{2x-x^2}}\,dx}_{\text{type II}}$$

$$= \left(\lim_{c\to 0^+} \int_c^1 \frac{1}{\sqrt{2x-x^2}}\,dx\right) + \left(\lim_{s\to 2^-} \int_1^s \frac{1}{\sqrt{2x-x^2}}\,dx\right)$$

$$= \lim_{c\to 0^+} \left(\arcsin(x-1)\Big|_c^1\right) + \lim_{s\to 2^-}\left(\arcsin(x-1)\Big|_1^s\right)$$

$$= \lim_{c\to 0^+}\left(0 - \underbrace{\arcsin(c-1)}_{\to -\frac{\pi}{2}}\right) + \lim_{s\to 2^-}\left(\underbrace{\arcsin(s-1)}_{\to \frac{\pi}{2}} - 0\right)$$

$$= \frac{\pi}{2} + \frac{\pi}{2} = \pi \qquad \text{Convergent}$$

$$\int \frac{1}{\sqrt{2x-x^2}}\,dx = \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx = \arcsin(x-1) + k$$

Example: $$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{1} \frac{1}{1+x^2}\,dx + \int_{1}^{\infty} \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to-\infty} \int_{R}^{1} \frac{1}{1+x^2}\,dx + \lim_{R\to\infty} \int_{1}^{R} \frac{1}{1+x^2}\,dx$$

$$= \lim_{R\to-\infty} \left(\arctan(1) - \arctan(R)\right) + \lim_{R\to\infty} \left(\arctan(R) - \arctan(1)\right)$$

$$= \cancel{\frac{\pi}{4}} - \frac{-\pi}{2} + \frac{\pi}{2} - \cancel{\frac{\pi}{4}} = \pi$$

p-integrals: let $0 < a < \infty$

$$\int_{a}^{\infty} \frac{1}{x^p}\,dx = \lim_{R\to\infty} \int_{a}^{R} \frac{1}{x^p}\,dx = \begin{cases} \text{if } p=1, & \lim\limits_{R\to\infty} \ln(R) - \ln(a) = +\infty \\ \text{if } p\neq 1, & \lim\limits_{R\to\infty} \left.\dfrac{x^{-p+1}}{-p+1}\right|_{a}^{R} = \lim\limits_{R\to\infty} \dfrac{R^{1-p} - a^{1-p}}{1-p} \end{cases}$$

$$= \begin{cases} \text{if } p>1, & \dfrac{-a^{1-p}}{1-p} \quad \text{convergent} \\ \text{if } p\leq 1, & +\infty \quad \text{divergent} \end{cases}$$

$$\int_{0}^{a} \frac{1}{x^p}\,dx = \lim_{c\to 0^+} \int_{c}^{a} \frac{1}{x^p}\,dx = \underset{\cdots\cdots}{\text{exercise!}} = \begin{cases} \text{if } p\geq 1, & +\infty \quad \text{divergent} \\ \text{if } p<1, & \dfrac{a^{1-p}}{1-p} \quad \text{convergent} \end{cases}$$

Fact: If f is continuous on $[a,b]$, then

$$\int_{a}^{b} f(x)\,dx = \lim_{c\to b^-} \int_{a}^{c} f(x)\,dx = \lim_{c\to a^+} \int_{c}^{b} f(x)\,dx$$

Comparison test for integrals let $-\infty \leq a < b \leq +\infty$ and let f, g be continuous functions on (a,b) with $0 \leq f(x) \leq g(x)$. Then

- If $\int_{a}^{b} g(x)\,dx$ converges, then $\int_{a}^{b} f(x)\,dx$ converges.
- If $\int_{a}^{b} f(x)\,dx$ diverges, then $\int_{a}^{b} g(x)\,dx$ diverges

Limit comparison test for integrals: let f, g be continuous on $[a,\infty)$ and $0 \le f(x), g(x)$

and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = k$. Then

- If $k = 0$, then $\int_a^\infty g(x)\,dx$ convergent $\Rightarrow \int_a^\infty f(x)\,dx$ convergent
- If $0 < k < +\infty$, then $\int_a^\infty g(x)\,dx$ convergent $\Leftrightarrow \int_a^\infty f(x)\,dx$ convergent
- If $k = +\infty$, then $\int_a^\infty f(x)\,dx$ convergent $\Rightarrow \int_a^\infty g(x)\,dx$ convergent

Fact: This statement has variations for improper integrals of type II where one replaces ∞ by the point at which we possibly have unboundedness. We shall not list these variations here explicitly.

Example: Determine whether or not $\int_2^\infty \frac{\sin x + x}{x^3+x}\,dx$ converges.

Solution: Note that $0 \le \frac{\sin x + x}{x^3+x} \le \frac{1+x}{x^3} = \frac{1}{x^3} + \frac{1}{x^2} \le \frac{2}{x^2}$ for all $2 \le x < \infty$

Since $\int_2^\infty \frac{2\,dx}{x^2}$ converges, by the CT, $\int_2^\infty \frac{\sin x + x}{x^3+x}\,dx$ converges.

Example: Determine whether or not $\int_1^\infty \frac{1}{x^{1+e^{-x}}}\,dx$ converges.

Solution: Note that $0 \le \frac{1}{x}, \frac{1}{x^{1+e^{-x}}}$ for $x \ge 1$ and moreover,

$$\lim_{x\to\infty} \frac{\cancel{\frac{1}{x}}}{\frac{1}{\cancel{x}^{1+e^{-x}}}} = \lim_{x\to\infty} x^{e^{-x}} = \lim_{x\to\infty} e^{e^{-x}\ln x} = e^{\lim_{x\to\infty} e^{-x}\ln x} = e^0 = 1.$$

$$\boxed{\lim_{x\to\infty} e^{-x}\ln x = \lim_{x\to\infty} \frac{\ln x}{e^x} \left(\frac{\infty}{\infty}\right) \overset{L'H}{=} \lim_{x\to\infty} \frac{\frac{1}{x}}{e^x} = 0}$$

Since $\int_1^\infty \frac{1}{x}\,dx = +\infty$ is divergent,

by the LCT, $\int_1^\infty \frac{1}{x^{1+e^{-x}}}\,dx$ is divergent.

Example: Determine whether or not $\int_0^\infty \frac{1}{x^x}\,dx$ converges.

Solution: We have $\int_0^\infty \frac{1}{x^x}\,dx = \int_0^2 \frac{1}{x^x}\,dx + \int_2^\infty \frac{1}{x^x}\,dx$. Observe that

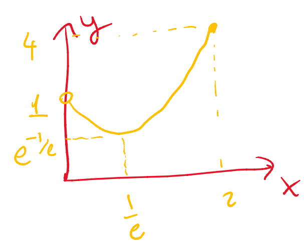
$$\frac{1}{x^x} \le \frac{1}{e^{-1/e}} \quad \text{for all } 0 < x \le 2.$$

$f(x) = x^x = e^{x\ln x}$

$f'(x) = e^{x\ln x}\cdot\left(\ln x + x\cdot\frac{1}{x}\right) = x^x(\ln x + 1) \Rightarrow f'(x) = \frac{1}{e}$

x	0	$\frac{1}{e}$	2
$f'(x)$	$-$	0	$+$
$f(x)$	$\searrow$		$\nearrow$

$\lim_{x\to 0^+} x^x = 1$ was done in class



$f\left(\frac{1}{e}\right) = \frac{1}{e^{1/e}}$

$f(x) = x^x \ge e^{-\frac{1}{e}}$

Since $\int_0^2 \frac{1}{e^{-1/e}}\,dx = \frac{2}{e^{-1/e}}$ is convergent by the CT, $\boxed{\int_0^1 \frac{1}{x^x} \text{ is convergent}}$

Also, $\frac{1}{x^x} \le \frac{1}{x^2}$ for all $x \ge 2$.

Since $\int_2^\infty \frac{1}{x^2}\,dx$ is convergent, by the CT, $\boxed{\int_2^\infty \frac{1}{x^x} \text{ is convergent}}$. Thus

$\int_0^\infty \frac{1}{x^x}\,dx$ is convergent.